Self-organization of two-dimensional incompressible viscous flow in a friction-free box

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The process by which self-organization occurs for two-dimensional incompressible viscous flow in a friction-free box is investigated theoretically with the use of numerical simulations. It is shown by analytical and numerical eigenfunction spectrum analyses that two basic processes for the self-organization are the spectrum transfer by nonlinear couplings and the selective dissipation among the eigenmodes of the dissipative operator, and they yield spectrum accumulation at the lowest eigenmode. The third important process during nonlinear self-organization is an interchange between the dominant operators, which has hitherto been overlooked in conventional self-organization theories and which leads to a final self-similar coherent structure with the lowest eigenmode of the dissipative operator. [S1063-651X(96)12109-7]

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Theories and numerical investigations have been described for self-organization in three-dimensional (3D) magnetohydrodynamic (MHD) plasmas [1,2], two-dimensional (2D) MHD plasmas [3], 2D incompressible viscous fluids [4–6], and solitons described by the Korteweg–de Vries (KdV) equation [7–9]. The theories of self-organization in 2D incompressible viscous fluids [4–6] are complex and are not unified yet. On the other hand, if we start from a simple definition for the self-organized state based on autocorrelations, then the self-organized state so derived does depend explicitly on the dissipative operator of the dynamical system [10–12]. Some simulations on 3D MHD plasmas [12] have reported data that show the dependence of the self-organized state on the profile of the dissipation parameters. The theory of self-organization in [10,11] will be useful to unify the physical pictures of self-organization in the 2D incompressible viscous fluids [4–6]. In this paper, we present analytical and numerical investigations of the process by which self-organization occurs for a 2D incompressible viscous flow in a friction-free box.

We apply here the self-organization theory by one of the authors (Y.K.) [10,11], which is based on the realization of the coherent sturcture with the minimum change rate of autocorrelations for their instantaneous values, to 2D incompressible viscous fluids. Taking the curl of the Navier-Stokes equation, we use the following vorticity representation:

$$\partial \boldsymbol{\omega}/\partial t = -(\mathbf{u}\cdot\nabla)\boldsymbol{\omega} + \nu\nabla^2\boldsymbol{\omega}, \tag{1}$$

where $\mathbf{u}$ is the fluid velocity, $\boldsymbol{\omega} = \nabla\times\mathbf{u}$ is the vorticity, ν is the kinematic viscosity, and $\nabla\cdot\mathbf{u} = 0$. The global autocorrelation $W_{\boldsymbol{\omega}}$ of $\boldsymbol{\omega}$ and its dissipation rate $\partial W_{\boldsymbol{\omega}}/\partial t$ are written, respectively, as $W_{\boldsymbol{\omega}} = \int \boldsymbol{\omega}\cdot\boldsymbol{\omega} dV$ and $\partial W_{\boldsymbol{\omega}}/\partial t = -2\int \boldsymbol{\omega}\cdot(\nu\nabla\times\nabla\times\boldsymbol{\omega}) dV$, where $\nabla\cdot\boldsymbol{\omega} = 0$ is used. Using variational calculus to find the self-organized state for which the rate of change for the autocorrelations of instantaneous values is minimum, and defining a functional F with the use of a Lagrange multiplier α as $F \equiv -\partial W_{\boldsymbol{\omega}}/\partial t - \alpha W_{\boldsymbol{\omega}}$, we obtain the following Euler-Lagrange equation from $\delta F = 0$ for the self-organized state $\boldsymbol{\omega}^*$ [10,11]:

$$\nabla\times\nabla\times\boldsymbol{\omega}^* = (\alpha/2\nu)\boldsymbol{\omega}^*. \tag{2}$$

When we work in the velocity representation of the Navier-Stokes equation, we obtain the same type of Euler-Lagrange equation for the velocity $\mathbf{u}^*$ at the self-organized state, as $\nabla\times\nabla\times\mathbf{u}^* = (\alpha/2\nu)\mathbf{u}^*$ [10,11]. Using the same procedure in [10,11], we obtain the following: $W^*_{\boldsymbol{\omega}} = e^{-\alpha t}W^*_{\boldsymbol{\omega}R} = e^{-\alpha t}\int[\boldsymbol{\omega}^*_R(\mathbf{x})]^2 dV$, $\boldsymbol{\omega}^* = \boldsymbol{\omega}^*_R(\mathbf{x})e^{-(\alpha/2)t}$, $W^*_{\mathbf{u}} = e^{-\alpha t}W^*_{\mathbf{u}R} = e^{-\alpha t}\int[\mathbf{u}^*_R(\mathbf{x})]^2 dV$ and $\mathbf{u}^* = \mathbf{u}^*_R(\mathbf{x})e^{-(\alpha/2)t}$. Here $\boldsymbol{\omega}^*_R(\mathbf{x})$ and $\mathbf{u}^*_R(\mathbf{x})$ denote the eigensolutions for $\boldsymbol{\omega}^*$ and $\mathbf{u}^*$ for given boundary values, which are supposed to be realized at the state with the minimum dissipation rate. We find from these equations that the eigenfunction of $\boldsymbol{\omega}^*(\mathbf{x})$ [or $\mathbf{u}^*(\mathbf{x})$] for the dissipative operator $-\nu\nabla\times\nabla\times\boldsymbol{\omega}$ [or $-\nu\nabla\times\nabla\times\mathbf{u}$] constitutes the self-organized and self-similar decay phase during the time evolution. From $\delta^2 F \geqslant 0$, we obtain the following associated eigenvalue problems for critical perturbations $\delta\boldsymbol{\omega}$ that make $\delta^2 F$ vanish, and the condition $0 < \alpha \leqslant \alpha_1$ for the state with the minimum dissipation rate [10,11]: $\nabla\times\nabla\times\delta\boldsymbol{\omega}_k - \lambda_k^2\delta\boldsymbol{\omega}_k = \mathbf{0}$, and $\nabla\times\nabla\times\delta\mathbf{u}_k - \lambda_k^2\delta\mathbf{u}_k = \mathbf{0}$. Here, $\lambda_k^2 \equiv \alpha_k/2\nu$, α_k and λ_k are the eigenvalues, $\delta\boldsymbol{\omega}_k$ denotes the eigensolutions, α_1 is the smallest positive eigenvalue, the boundary conditions are $\delta\boldsymbol{\omega}_w\cdot d\mathbf{S} = 0$, and the subscript w denotes the value at the boundary wall. When we work inside a square friction-free box, Eq. (2) becomes equivalent to the associated eigenvalue problems shown above. Therefore, the decay constant α of the autocorrelation $W^*_{\boldsymbol{\omega}}$ (or $W^*_{\mathbf{u}}$) at the self-organized state is equal to the smallest eigenvalue α_1 $(=2\nu\lambda_1^2)$, and $\boldsymbol{\omega}^*$ coinsides with the lowest eigensolution $\delta\boldsymbol{\omega}_1$.

We now analytically describe a physical picture for the self-organization process by using an eigenfunction spectrum analysis associated with the dissipative operator [10,11]. Owing to the self-adjoint property of the present dissipative operator [10,11], the eigenfunctions $\mathbf{a}_k$ for the associated eigenvalue problems form a complete orthogonal set and the appropriate normalization is written as $\int \mathbf{a}_k\cdot(\nabla\times\nabla\times\mathbf{a}_j) dV = \int \mathbf{a}_j\cdot(\nabla\times\nabla\times\mathbf{a}_k) dV = \lambda_k^2\int \mathbf{a}_j\cdot\mathbf{a}_k dV = \lambda_k^2\delta_{jk}$, where $\nabla\times\nabla\times\mathbf{a}_k - \lambda_k^2\mathbf{a}_k = \mathbf{0}$ is used. For the present case inside a square friction-free box with edge length 1 in the

(x,y) plane, the normalized orthogonal eigensolutions of $\mathbf{a}_{\omega k}$ for the vorticity and $\mathbf{a}_{uk}$ for the velocity are obtained as follows:

$$\mathbf{a}_{\omega k}=2\sin(l_k\pi x)\sin(m_k\pi y)\mathbf{k}, \tag{3}$$

$$\mathbf{a}_{uk}=\frac{2}{\sqrt{l_k^2+m_k^2}}(m_k\sin(l_k\pi x)\cos(m_k\pi y)\mathbf{i} - l_k\cos(l_k\pi x)\sin(m_k\pi y)\mathbf{j}), \tag{4}$$

where $\lambda_k^2 = \pi^2(l_k^2+m_k^2)$, $l_k \geqslant 1$, $m_k \geqslant 1$, l_k and m_k are mode numbers in x and y directions, respectively, and $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the unit vectors in x,y, and z directions, respectively. Here, $\boldsymbol{\nabla}\times\mathbf{a}_{uk} = \pi\sqrt{l_k^2+m_k^2}\mathbf{a}_{\omega k}$. The distributions of $\boldsymbol{\omega}$ and $\mathbf{u}$ at each instant can then be expanded with the use of these normalized orthogonal eigenfunctions $\mathbf{a}_{\omega k}$ and $\mathbf{a}_{uk}$, as follows: $\boldsymbol{\omega} = \Sigma_{k=1}^{\infty} c_{\omega k}\mathbf{a}_{\omega k}$, and $\mathbf{u} = \Sigma_{k=1}^{\infty} c_{uk}\mathbf{a}_{uk}$, where $\pi\sqrt{l_k^2+m_k^2}c_{uk} = c_{\omega k}$, and the spectra of $c_{\omega k}$ and c_{uk} ($k = 1, 2, \ldots$) depend now on time t. Substituting these equations of $\boldsymbol{\omega}$ and $\mathbf{u}$ into Eq. (1), we obtain the following:

$$\sum_{k=1}^{\infty}\frac{\partial c_{\omega k}}{\partial t}\mathbf{a}_{\omega k} = -\left(\sum_{k=1}^{\infty} c_{uk}\mathbf{a}_{uk}\cdot\boldsymbol{\nabla}\right)\left(\sum_{k=1}^{\infty} c_{\omega k}\mathbf{a}_{\omega k}\right) - \sum_{k=1}^{\infty}\nu\lambda_k^2 c_{\omega k}\mathbf{a}_{\omega k}. \tag{5}$$

With the use of Eqs. (3) and (4), the nonlinear coupling terms and the dissipative terms in Eq. (5) are written, respectively, as follows:

$$-\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(c_{ui}\mathbf{a}_{ui}\cdot\boldsymbol{\nabla})c_{\omega j}\mathbf{a}_{\omega j} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{c_{\omega i}c_{\omega j}}{l_i^2+m_i^2}\{(l_i m_j - m_i l_j)[\sin[(l_i+l_j)\pi x]\sin[(m_i+m_j)\pi y] - \sin[(l_i-l_j)\pi x]\times\sin[(m_i-m_j)\pi y]] + (l_i m_j + m_i l_j)[\sin[(l_i+l_j)\pi x]\sin[(m_i-m_j)\pi y] - \sin[[(l_i-l_j)\pi x]\sin[(m_i+m_j)\pi y]]\}\mathbf{k}. \tag{6}$$

$$-\sum_{k=1}^{\infty}\nu\lambda_k^2 c_{\omega k}\mathbf{a}_{\omega k} = -\sum_{k=1}^{\infty} 2\nu\pi^2(l_k^2+m_k^2)c_{\omega k}\sin(l_k\pi x)\sin(m_k\pi y)\mathbf{k}. \tag{7}$$

We see the following three basic processes from Eqs. (6) and (7): (A) The nonlinear coupling terms induce the spectrum transfers to both the higher and the lower eigenmodes of $(l_i\pm l_j, m_i\pm m_j)$, while the eigenmode (l_i, m_i) does not have the nonlinear coupling with itself. (B) The dissipative terms yield the selective dissipation among the eigenmodes, i.e., the higher spectral components dissipate more rapidly in proportion to the decay constant of $\nu\pi^2(l_k^2+m_k^2)$. (C) After a long term dissipation with the spectrum transfers and the selective dissipation, spectral components $c_{\omega k}$ will become smaller so that $|c_{\omega i}c_{\omega j}(m_i l_j \pm l_i m_j)/(l_i^2+m_i^2)| < 2\nu\pi^2(l_k^2+m_k^2)c_{\omega k}$ even to the lowest eigenmode (1,1) [cf. the right hand sides of Eqs. (6) and (7)], and the dominant operator changes consequently from the nonlinear coupling terms to the dissipative terms. Due to (A)–(C), the lowest eigenmode persists to the end.

We now show some typical computational results. We solve Eq. (1) in a dimensionless unit, inside a square friction-free box in the x,y plane, with edge length 1. The fluid velocity $\mathbf{u} = \boldsymbol{\nabla}\psi\times\mathbf{k}$, where the stream function $\psi = \psi(x,y,t)$ is independent of z, as are all other field variables. The vorticity $\boldsymbol{\omega} = \nabla\times\mathbf{u} = \omega\mathbf{k}$, and the relation between ω and ψ is given by $\nabla^2\psi = -\omega$. We solve the hyperbolic equation of Eq. (1) by using a scheme, named the KOND (kernel optimum nearly analytical discretization algorithm) scheme [14], which has high numerical accuracy and stabitilty. We use the SOR (successive over-relaxation) scheme to solve the elliptic type equation $\nabla^2\psi = -\omega$. Numerical procedures at each time step are as follows; (1) solve $\nabla^2\psi = -\omega$ by the SOR scheme to get new values of ψ, (2) get new values of $\mathbf{u}$ from the new ψ, (3) solve Eq. (1) by the KOND scheme to get new values of $\boldsymbol{\omega}$, and (4) go to (1) for the next time step. The boundary conditions at the friction-free wall are given by $\psi_{\mathbf{w}} = 0$ and $\omega_{\mathbf{w}} = 0$, where the subscript w denotes the values at the boundary wall. The simulation domain is implemented on a (101×101) point grid. The time step is $\Delta t = 0.0001$. The kinematic viscosity ν can, in the dimensionless units, be interpreted as the reciprocal of a Reynolds number R based on unit length and a unit initial rms velocity, i.e., $R = \nu^{-1}$. We show here one of the typical results of simulations for cases with $R = 500$, whose initial flow structures are different with each other and do not contain the lowest eigenmode (1,1). In these cases, since the smallest eigenvalue corresponding to the eigenmode of (1,1) is $\lambda_1 = \sqrt{2}\pi$, the theoretical decay constant α_1 $(=2\nu\lambda_1^2)$ of the autocorrelation $W_{\mathbf{u}}^*$ at the self-organized state has the same value of 0.7896×10^{-1}, which was compared with the simulation results.

Figure 1 shows the typical time evolution of the vorticity structure during the self-organization process, which starts from an initial flow given by superposition of two eigenmodes of (2,4) and (1,5) with the use of Eq. (4) for the velocity. In earlier phases, the nonlinear process is seen to change the initial simple structure of vorticity into the more complicated structure with small scale deformations. Rotating clockwise around the center of the box (cf. the vorticity

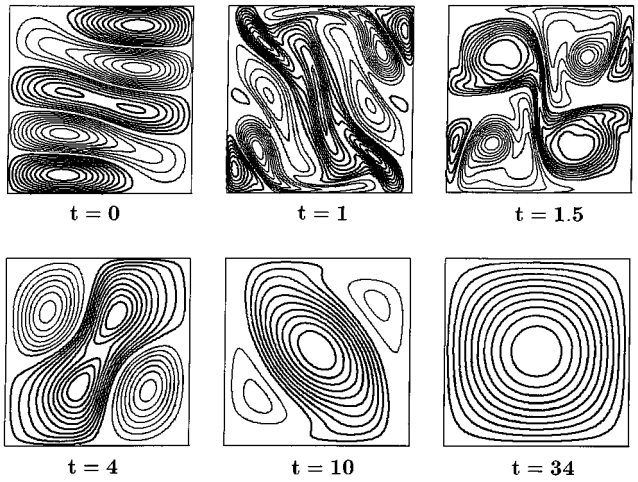

FIG. 1. Typical time evolution of vorticity structures during self-organization. The initial flow at $t = 0$ is given by superposition of two eigenmodes of (2,4) and (1,5) with the use of Eq. (4) for the velocity. The bold and the thin lines show contour plots of positive vorticity and those of negative one, respectively. The height of contours is normalized by the maximum absolute value of either the positive or the negative vorticity in each figure.

contours at $t = 1$, 1.5, 4, and 10), the two large positive vorticity centers at $t = 1.5$ merge gradually into the larger structure with one positive center at $t = 10$. The two outside negative vorticity centers gradually vanish, and the simplest structure with one positive vorticity center finally persists to the end.

In order to check the physical picture for the self-organization process described analytically in the preceding section, we performed numerical eigenfunction spectrum analysis for the simulation results shown in Fig. 1. Multiplying the simulation data of vorticity by the normalized orthogonal eigensolutions $\mathbf{a}_{\omega k}$ of Eq. (3) and integrating the results over the square box, we obtain numerically the spectral components of $c_{\omega k}$ at each time. Figure 2 shows the time evolution of the resultant spectral components of vorticity, which are obtained from the simulation data shown in Fig. 1. The vorticity spectrum at $t = 0$ is shown to have only two spectral components of (2,4) and (1,5) that correspond to the initial flow with the relation $\nabla \times \mathbf{a}_{uk} = \pi \sqrt{l_k^2 + m_k^2}\, \mathbf{a}_{\omega k}$. We find from the spectrum at $t = 1$ that the nonlinear process yields the spectrum transfer toward both the higher and the lower spectral eigenmodes, in other words, it yields both the normal and the inverse cascades. We recognize from the time evolution of spectra after $t = 1$ that the higher spectral components dissipate more rapidly and the spectrum transfer toward the lower eigenmode region yields gradually spectrum accumulation at the lowest eigenmode of (1,1) which persists to the end, as is shown by the spectrum at $t = 34$. It should be noted here that the eigenmode of (1,1) was not contained in the initial flow at $t = 0$, but has been induced nonlinearly during the self-organization process.

Figure 3 shows the time dependence of the flow energy E for the case of Fig. 1, where E is defined here by $E = \int \mathbf{u} \cdot \mathbf{u} dV$ and is equal to the global autocorrelation $W_{\mathbf{u}}$ with respect to the velocity $\mathbf{u}$. After a rapid decay lasting until around $t \sim 12$, the decay rate of E is seen to become almost constant. At around $t = 25$, the decay constant has a value of 0.791×10^{-1}, which agrees very well with the theoretical decay constant of $\alpha_1 = 2\nu\lambda_1^2 = 0.7896 \times 10^{-1}$.

Without dependence on different initial flow structures, simulations for other cases yielded similar results with the self-organization process, in the same way as was shown in Figs. 1–3.

In conclusion, applying the self-organization theory by one of the authors (Y.K.) [10,11] to the 2D incompressible viscous fluids, we have shown that the lowest eigensolution of Eq. (2) is predicted to be the self-organized state and the theoretical decay constant α of the autocorrelation $W_{\boldsymbol{\omega}}^*$ (or $W_{\mathbf{u}}^*$) is equal to the smallest eigenvalue α_1 $(= 2\nu\lambda_1^2)$. Using the eigenfunction spectrum analysis associated with the dis-

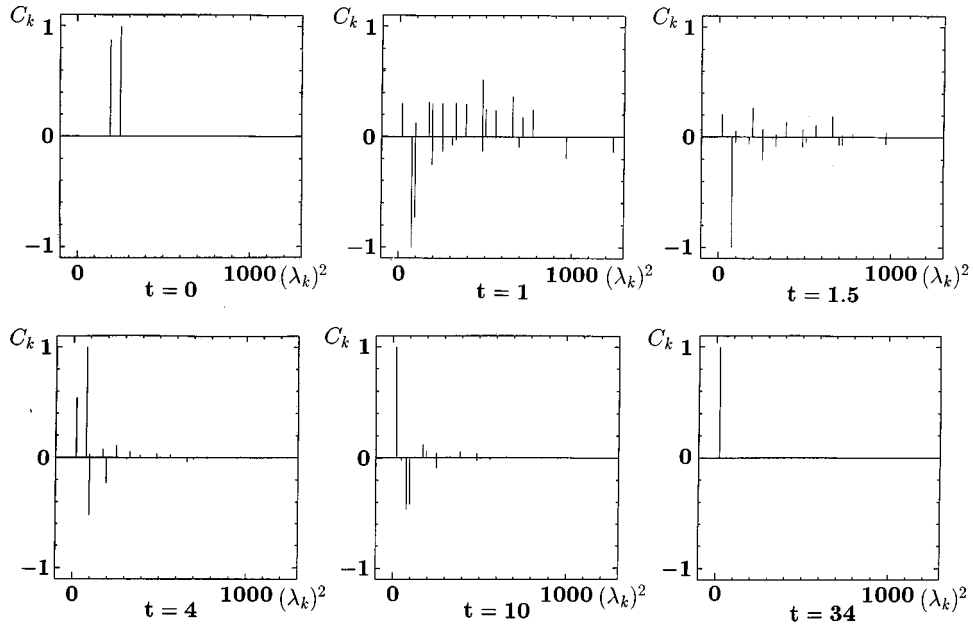

FIG. 2. Time evolution of spectral components of vorticity during self-organization, obtained from the simulation data shown in Fig. 1. Horizontal scale is given by the square of spectral eigenvalues $\lambda_k^2 = \pi^2(l_k^2 + m_k^2)$ for eigenmodes (l_k, m_k). Vertical scale is normalized by the maximum absolute value of either the positive or the negative spectral components $c_{\omega k}$ in each figure.

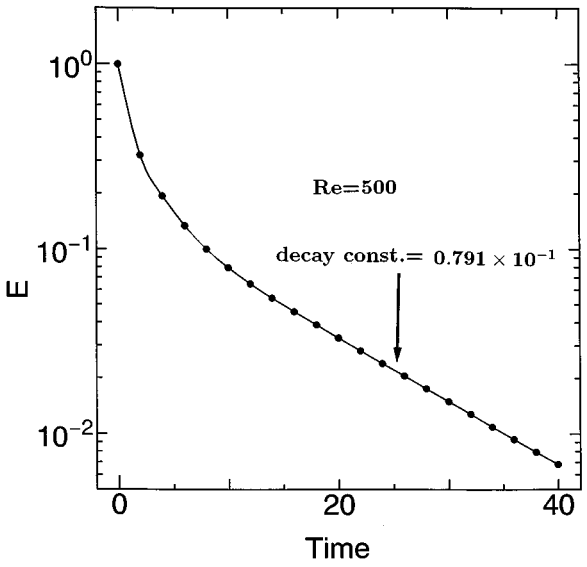

FIG. 3. Time dependence of the flow energy E, defined by $E = \int \mathbf{u}\cdot\mathbf{u}\mathrm{d}V$, for the case of Fig. 1. The numerical value of the decay constant at around $t = 25$ is 0.791×10^{-1}.

sipative operator [10,11], we have analytically described a physical picture for the self-organization process [from Eq. (3) to Eq. (7)], and have clarified the three basic processes of (A)–(C) shown after Eqs. (6) and (7). In order to demonstrate the self-organization process predicted by the theory, we have presented the typical results of numerical simulations, whose initial flow structures are simple but do not contain the lowest eigenmode (1,1). It has been shown clearly by the numerical eigenfunction spectrum analysis that the nonlinear process leads finally to the simplest structure with the lowest eigenmode (1,1), without dependence on the different initial structures. After the initial rapid decay, the numerical decay constant of the flow energy E has been shown to agree very well with the theoretical one given by $\alpha_1 = 2\nu\lambda_1^2$.

The analytical and the numerical investigations for the self-organization presented here may lead to the following physical picture for the self-organization process, which is also common to other self-organization processes in solitons described by the KdV equation with a viscous dissipation term [9] and in 3D resistive MHD plasmas [13]: (1) The nondissipative nonlinear operator induces the spectrum transfer toward both the higher and the lower eigenmode regions of the dissipative operator (i.e., both of the normal and the inverse cascades). (The spectrum transfer toward the lower eigenmode region yields spectrum accumulation at the lowest eigenmode. The spectrum transfer to the higher eigenmode region results in the spread of the spectrum to the infinity.) (2) At the same time, the dissipative operator yields the selective dissipation among the eigenmodes of the dissipative operator, i.e., the higher spectral components dissipate more rapidly with decay constants of $\nu\lambda_k^2$ (3) In the later phase of self-organization, there occurs an interchange between the dominant operators from the nondissipative nonlinear operator to the dissipative operator, and the lowest eigenmode consequently persists to the end as a final self-similar coherent structure.

The study of the self-organization presented here suggests that the principle of the minimum dissipation rate of enstrophy ($W_{\boldsymbol{\omega}} = \int \boldsymbol{\omega}\cdot\boldsymbol{\omega}\mathrm{d}V$) can be used for the theory of self-organization as well as the principle of the minimum dissipation rate of energy ($E = \int \mathbf{u}\cdot\mathbf{u}\mathrm{d}V$). However, the more essential physics contained fundamentally is the principle of the minimum dissipation rate of autocorrelations ($W_{ii} = \int q_i\cdot q_i\mathrm{d}V$) in the dynamical systems [11].

In the case of the periodic boundary condition (such as reported in [4–6]), the eigensolution of $\mathbf{a}_{\omega k}$ for the vorticity is given by $\mathbf{a}_{\omega k} = \exp[i2\pi(l_k x + m_k y)]\mathbf{k}$, and the lowest eigenmode of $[(1,0)+(0,1)]$ is the self-organized state predicted by the present theory. Results of our numerical eigenfunction spectrum analysis for the case of the periodic boundary condition will be reported elsewhere.

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